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ON THE AXIOM OF SEPARABILITY

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Abstract This work studies axiomatic systems for additive and separable preference structures on a product set, and correspondingly explores roles of different separability conditions in utility representations. We show that the Thomsen condition is a necessary and sufficient condition for an independent preference structure being additive on a general two-dimensional domain. Moreover, a theorem is developed so as to identify a large class of separable preference relations that can in effect admit additive utility representations, *videlicet*, a separable preference structure on a two-dimensional domain is essentially additive, if and only if its separability rule can represent some additive preference relation on the real plane.

Keywords: preference relation, utility function, separability rule, additivity, separability, Thomsen condition, 3-web.
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1. INTRODUCTION

Consider a general domain Σ, on which a rational preference relation can be defined. Suppose Σ can be formalized as a Cartesian product set,

$$\Sigma = \prod_{i=1}^{n} X_i \quad (n \in \mathbb{N}).$$

Example 1.1. *Let $\Sigma = X_1 \times X_2 \times \cdots \times X_n$, where $X_i \subseteq \mathbb{R}$ for all $i \in \{1, 2, \ldots, n\}$, and $n \in \mathbb{N}$. Here, $\sigma \in \Sigma$ can be a bundle of n different commodities, or an array of n independent stimulus variables affecting a certain attribute of an observable object.*

Let $\Sigma = X \times X$, where X is a set of sure prospects, then any $(x_1, x_2) \in \Sigma$ can represent an uncertain prospect of being x_1 and x_2 with some allocated probabilities.

Let $\Sigma = X \times T$, where X is a set of events, and T is a time domain, then any $(x, t) \in \Sigma$ denotes a realization x at a specific time $t \in T$.

Let $\Sigma = X \times Q$, where $X \subseteq \mathbb{R}^+$ and $Q \subseteq [0, 1]$, then any $(x, q) \in \Sigma$ can represent such a gambling that a gambler wins x and loses $\sqrt{x}$ with probability q and $1 - q$, respectively.

We will then study rational preferences on such product sets, and in particular, be interested in such properties as separability and additivity per se in their utility (value) representations. Such investigations can be found in the books by Fishburn [4, Chapter 4 & 5], Krantz et al. [9, Chapter 6 & 7], Roberts [11, Chapter 5], and

Wakker [13, Chapter II & III], as well as the contributions by Gorman [7], Debreu [3], Luce and Tukey [10], Fishburn [5], Karni and Safra [8], Bouyssou and Marchant [2], and many other authors.

A utility function $u : \Sigma \to \mathbb{R}$ is called an *additive conjoint representation* of a preference relation on $\Sigma = \prod_{i=1}^{n} X_i$, if for all $(x_1, x_2, \ldots, x_n) \in \Sigma$

$$u(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} u_i(x_i), \tag{1}$$

where $u_i : X_i \to \mathbb{R}$ for all i. A preference relation that admits an additive conjoint representation is then called *additively separable* or just *additive*.

More generally, a utility function $u : \Sigma \to \mathbb{R}$ is called a *separable representation* of a preference relation on $\Sigma = \prod_{i=1}^{n} X_i$, if there is a function $v : \mathbb{R}^n \to \mathbb{R}$, such that for all $(x_1, x_2, \ldots, x_n) \in \Sigma$

$$u(x_1, x_2, \ldots, x_n) = v(u_1(x_1), u_2(x_2), \ldots, u_n(x_n)), \tag{2}$$

where $u_i : X_i \to \mathbb{R}$ for all i. The aggregation operator v in a separable utility function is called a *separability rule*. As a special case, the summation operator in an additive utility function is an additive separability rule. We shall call a preference relation that admits a separable representation *generally separable* or just *separable*.

The structure of this work is arranged as follows. Section 2 aims to connect the existing studies to our proposed investigations. Four distinct axiomatic systems, which are respectively based on the solvability, the Thomsen condition, the double cancellation condition, and the stationarity, are constructed so that they can all sufficiently generate additive preference structures, with differential capabilities yet.

In Section 3, the Thomsen condition on the plane is studied from the perspective of web geometry. We first make a more intuitive definition for the Thomsen condition by employing a transformation operator in the web-covered plane. We next show a critical lemma stating that a preference relation satisfies the Thomsen condition if and only if its corresponding 3-web on the plane is hexagonal.

In Section 4, we could then show that the Thomsen condition is also a necessary condition for the additivity of a preference structure on a general domain. And finally, we state a theorem to provide an identification rule to check whether a separable preference structure is actually additive.

2. AXIOMATIC SYSTEMS

In this section, we will summarize some developed axiomatic systems that can sufficiently ensure separable preference structures. Before stating these results, we have to make some formal definitions.

Let $\gtrsim$ denote a binary relation on the product set $\Sigma = \prod_{i=1}^{n} X_i$. $\gtrsim$ is called

1 *reflexive*, if $\sigma \gtrsim \sigma$ for all $\sigma \in \Sigma$,

2 *transitive*, if $\sigma_1 \gtrsim \sigma_2$ and $\sigma_2 \gtrsim \sigma_3$ imply $\sigma_1 \gtrsim \sigma_3$ for all $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$,

3 *complete*, if either $\sigma_1 \gtrsim \sigma_2$ or $\sigma_2 \gtrsim \sigma_1$ for all $\sigma_1, \sigma_2 \in \Sigma$.

A *preference relation* $\gtrsim$ on Σ is essentially a binary relation on the same Σ.

Definition 2.1. *A preference relation $\gtrsim$ on a domain Σ is rational, if it is reflexive, transitive, and complete.*

If Σ is countable, or Σ is uncountable but $\gtrsim$ is dense on it, there should be an isomorphism $u : \Sigma \to \mathbb{R}$ such that $\gtrsim$ on Σ can be preserved by $\geq$ on $\mathbb{R}$, or in terms of utility theory, there is an equivalent utility function $u : \Sigma \to \mathbb{R}$, such that $\sigma_1 \gtrsim \sigma_2$ if and only if $u(\sigma_1) \geq u(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$.

It would be helpful to make a division of $\sigma \in \Sigma$ especially when the dimension of Σ is greater than 3. Recall that $\sigma = (x_1, x_2, \ldots, x_n)$, for all i there is a subtuple $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ which belongs to $\prod_{j \neq i} X_j$. Let x_{-i} denote such an $(n-1)$-tuple, and moreover, set $x^-_{-i} = (x_1, x_2, \ldots, x_{i-1})$ and $x^+_{-i} = (x_{i+1}, \ldots, x_n)$, where $x^-_{-1}, x^+_{-n} \in \emptyset$. We can then write $x_{-i} = (x^-_{-i}, x^+_{-i})$, and hence $\sigma = (x^-_{-i}, x_i, x^+_{-i})$. Without putting any confusion, we shall simply write $\sigma = (x_i, x_{-i})$ for all i, just keeping in mind that the index should be always labeled in lexicographic order.

Suppose $\gtrsim$ is a rational preference on $\Sigma = \prod_{i=1}^n X_i$. If $\sigma_1 \gtrsim \sigma_2$ as well as $\sigma_2 \gtrsim \sigma_1$, we shall write $\sigma_1 \sim \sigma_2$, where $\sim$ is called the indifference relation corresponding to $\gtrsim$. Let $\gtrsim_i$ denote the rational preference on X_i induced from $\gtrsim$ on Σ, and let $\gtrsim_{-i}$ denote the rational preference on $\prod_{j \neq i} X_j$ induced from $\gtrsim$ on Σ. Here, $\gtrsim_i$ and $\gtrsim_{-i}$ are both said to be induced from $\gtrsim$, because their domains are conditional on Σ and they reflect part of the original $\gtrsim$. With regards their formal definitions, one can state them similarly like the above definition of a rational preference $\gtrsim$.

Definition 2.2. *A preference relation $\gtrsim$ on Σ is independent, if $(x_i, x_{-i}) \gtrsim (y_i, x_{-i})$ is equivalent to $x_i \gtrsim_i y_i$ for all $i \in \{1, 2, \ldots, n\}$.*

Definition 2.3. *A preference relation $\gtrsim$ on Σ is solvable, if $(x_i, x_{-i}) \gtrsim \sigma \gtrsim (y_i, x_{-i})$ implies there exists a $z_i \in X_i$ such that $(z_i, x_{-i}) \sim \sigma$ for all $i \in \{1, 2, \ldots, n\}$.*

We can then establish a somewhat restricted axiomatic system for an additive preference structure,

Theorem 2.1. *If a preference relation $\gtrsim$ is rational, independent, and solvable on $\Sigma = \prod_{i=1}^n X_i$, then it is additive, and admits an additive conjoint representation.*

Sketch of Proof. By the independence condition, $(x_i, x_{-i}) \gtrsim (y_i, x_{-i})$ if and only if $x_i \gtrsim_i y_i$ for all i. Since $\gtrsim$ is solvable on Σ, $\gtrsim_i$ is dense on X_i. Thus there exists a utility function $u_i : X_i \to \mathbb{R}$ representing $\gtrsim_i$. By the solvability condition, each x_i is essential, so for all $\sigma \in \Sigma$, we can have $(x_i, x_{-i}) \gtrsim \sigma \gtrsim (x_i, y_{-i})$. Then we can find a z_{-i} such that $\sigma \sim (x_i, z_{-i})$ by applying the solvability condition $n-1$ times. So there also exists a function $u_{-i} : \prod_{j \neq i} X_j \to \mathbb{R}$ representing $\gtrsim_{-i}$. We hence obtain $2n$

functions, $u_1, u_2, \ldots, u_n$, and $u_{-1}, u_{-2}, \ldots, u_{-n}$. Note that u_{-i} must be composed of all the u_j's for $j \neq i$, otherwise some x_j's would not be essential. The utility function $u : \Sigma \to \mathbb{R}$ representing $\succsim$ is thus determined by n functions, $u_1, u_2, \ldots, u_n$. Notice that the role of u_i in u_{-j} and that of u_j in u_{-i} is inverse to each other for all $i \neq j$, but the function form of u is constant, thus we must have $u_i + u_j$ in u. It therefore comes to us that $u = \sum_{i=1}^{n} u_i$, which completes the proof. ∎

Note that the condition of solvability is a rather strong axiom of separability, as any factor can be separated from the other $n - 1$ factors. Debreu [3] presented an axiomatic system with a weaker separability condition for preference relations on a two-dimensional domain. To follow this tradition and also to simplify our analysis, we will focus on two-dimensional domains in the remaining parts of this work.

Let $\Sigma = X_1 \times X_2$, where X_1 and X_2 are general topologically connected spaces, as was similarly assumed by Debreu [3].

Definition 2.4. *A preference relation $\succsim$ on $X_1 \times X_2$ satisfies the Thomsen condition, if $(x_1, y_2) \sim (y_1, x_2)$ and $(y_1, z_2) \sim (z_1, y_2)$ imply $(x_1, z_2) \sim (z_1, x_2)$ for all $x_1, y_1, z_1 \in X_1$ and $x_2, y_2, z_2 \in X_2$.*

If a preference relation $\succsim$ on $X_1 \times X_2$ is rational, independent, and satisfies the Thomsen condition, then it must have an additive structure, and could be represented by a certain additive utility function. Fishburn [4] generalized the two-dimensional domain in Debreu's axiomatic system to an n-dimensional domains for $n \geq 2$ finite, on which the general Thomsen condition can be defined by the relation between each pair of factors (see Theorem 5.5 in Fishburn [4], pp. 71–76). As a graphic illustration, Fig. 1 shows a preference relation satisfying the Thomsen condition on the real plane $\mathbb{R}^2$.

Definition 2.5. *A preference relation $\succsim$ on $X_1 \times X_2$ satisfies the double cancellation condition, if $(x_1, y_2) \succsim (y_1, x_2)$ and $(y_1, z_2) \succsim (z_1, y_2)$ imply $(x_1, z_2) \succsim (z_1, x_2)$ for all $x_1, y_1, z_1 \in X_1$ and $x_2, y_2, z_2 \in X_2$.*

Luce and Tukey [10] proposed a similar axiomatic system to Debreu's, in which the Thomsen condition is replaced by the double cancellation condition. The double cancellation condition is slightly stronger than the Thomsen condition, but again weaker than the solvability condition. So the range of preference relations that can be captured by the axiomatic system in Theorem 2.1 is smaller than that proposed by Luce and Tukey, and even much smaller than that proposed by Debreu.

As a special case of Debreu's axiomatic system, Fishburn and Rubinstein [6] stated an axiomatic system on the domain $X \times T$, where $X \subseteq \mathbb{R}$, and T is the time domain, either discrete as $\mathbb{Z}$ or dense on $\mathbb{R}$. According to Debreu's theorem, if the time preference $\succsim$ on $X \times T$ is rational, independent, and satisfies the Thomsen condition, there should be an additive conjoint representation $\mu : X \times T \to \mathbb{R}$, such that for all $(x, t) \in X \times T$

$$\mu(x, t) = \nu(x) + \varrho(t), \tag{3}$$

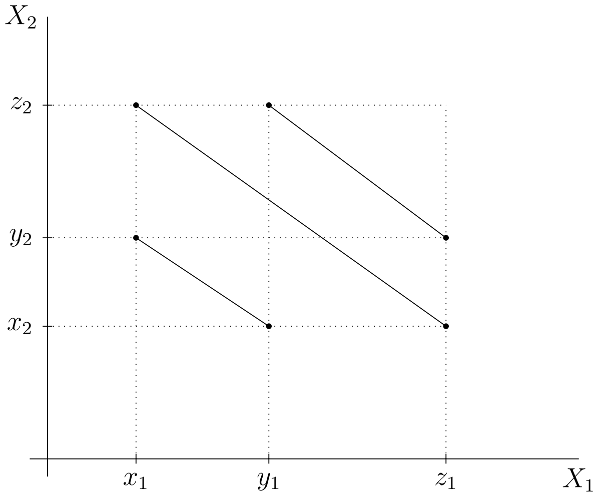

Fig. 1.: The Thomsen condition on the plane.

where $\nu : X \to \mathbb{R}$ and $\varrho : T \to \mathbb{R}$.

Recall that an ordinal utility representation must be invariant to any positively monotonic transformation, thus we can have another equivalent utility representation $u(x,t) = \exp(\mu(x,t))$, such that for all $(x,t) \in X \times T$

$$u(x,t) = \rho(t)v(x), \tag{4}$$

where $\rho = \exp(\varrho)$ and $v = \exp(\nu)$.

Definition 2.6. *A time preference $\succsim$ on $X \times T$ is stationary, if $(x,t) \sim (y,s)$ implies $(x, t+\tau) \sim (y, s+\tau)$ for all $x, y \in X$ and $t, s, t+\tau, s+\tau \in T$.*

Notice that the stationarity condition means that the induced preference $\succsim_t$ on the time dimension (as a naïve time preference) is linear and independent of the dimension x, so it is much stronger than the solvability condition.

As shown by Fishburn and Rubinstein [6], if the time preference $\succsim$ on $X \times T$ is rational, independent, and stationary, the additive utility representation will be quasi-linear, *videlicet*, $\varrho(t) = \alpha t$ for $\alpha < 0$. Thus $\mu(x,t) = \nu(x) + \alpha t$. Let $\beta = \mathrm{e}^{\alpha}$, then $\rho(t) = \exp(\alpha t) = \beta^t$, where $\beta \in (0,1)$, and hence the following exponentially scaled utility representation stands out:

$$u(x,t) = \beta^t v(x). \tag{4$'$}$$

Here, β is the discounting that measures the impatience of the time preference.

Summing up, we presented four axiomatic systems that are sufficient for additive representation of a preference relation on product sets; namely, the ones based on the stationarity condition, the solvability condition, the double cancellation condition, and the Thomsen condition, respectively. Among them, the Thomsen condition is the weakest one, while the stationarity condition is the strongest one. Once we plan to study the necessary condition for the additivity and separability of preference relations on product sets, we should first study the Thomsen condition. In case the Thomsen condition is not necessary for an additive or separable preference structure, then all the other stronger ones could not be, either.

3. THOMSEN CONDITION

In this section, we want to reconstruct the Thomsen condition by the theory of web geometry. We will again consider a two-dimensional domain $\Sigma = X_1 \times X_2$, where X_1 and X_2 are now general metric spaces.

Assume there always exists a diffeomorphism $f : \Sigma \to \mathbb{R}^2$ transforming Σ to an affine domain $\Sigma' = X \times Y$ of the plane $\mathbb{R}^2$. Under the diffeomorphism f, any point $(x_1, x_2) \in \Sigma$ is mapped uniquely to a corresponding point $(x, y) = f(x_1, x_2)$ in Σ', and also any smooth curve in Σ is transformed into a corresponding smooth curve in Σ'. A preference relation $\gtrsim$ on Σ is again a preference relation on Σ', as $\sigma_1 \gtrsim \sigma_2$ if and only if $f(\sigma_1) \gtrsim f(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$, where $f(\sigma_1), f(\sigma_2) \in \Sigma'$. (It means that an agent's preference relation as her subjective knowledge over Σ is shaped after her knowledge on the domain Σ and a generic transformation rule f on such a domain. So her preference relation can be kept over the domain $\Sigma' = f(\Sigma)$). Therefore, the Thomsen condition is satisfied by $\gtrsim$ on the domain Σ, if and only if it can be satisfied by the same preference relation on the affine domain Σ'.

In $\Sigma' = X \times Y$, through a given point (x, y) there exist infinitely many smooth curves, each of which can be determined by a mapping $g : \Sigma' \to \mathbb{R}$, such that $g(x, y)$ is a constant for all (x, y) on a same curve. Thus any mapping $g : \Sigma' \to \mathbb{R}$ actually determines a family of smooth curves on Σ'. Let γ denote a regular family of smooth curves on Σ'. Suppose the collection of all the regular families of smooth curves on Σ' can be expressed as

$$\Gamma(\Sigma') = \{\gamma_i : i \in I\},$$

where I denotes an index set. The mapping that determines the family $\gamma_i \in \Gamma(\Sigma')$ is denoted by $g_i : \Sigma' \to \mathbb{R}$ for all $i \in I$. We thus have an equivalent collection of mappings $\{g_i : i \in I\}$, in other words, $\Gamma(\Sigma') \simeq \{g_i : i \in I\}$.

Definition 3.1. $\{\gamma_1, \gamma_2, \gamma_3\}$ *is called a* 3-*web on* Σ', *if* $\gamma_1, \gamma_2, \gamma_3 \in \Gamma(\Sigma')$.

Since γ_i is solely determined by the mapping $g_i(x, y)$ for all $i \in I$, we can also use $\{g_1, g_2, g_3\}$ to denote a 3-web $\{\gamma_1, \gamma_2, \gamma_3\}$.

Example 3.1. *Suppose*

$$g_1(x, y) = x + y, \quad g_2(x, y) = x - y, \quad g_3(x, y) = xy,$$

then $\{x + y, x - y, xy\}$ *is a* 3-*web on* $X \times Y$.

Suppose

$$g_i(x, y) = x + a_i y \quad (i = 1, 2, 3),$$

where $a_1 \neq a_2 \neq a_3 \neq a_1$, then $\{x + a_1y, x + a_2y, x + a_3y\}$ is a linear 3-web on $X \times Y$, in which all the curves are lines.

A 3-web $\{\gamma_1, \gamma_2, \gamma_3\}$ is called *complete*, if

1 there exists only one curve in the family γ_i passing any point $\sigma \in \Sigma'$,

2 any two distinct curves in the same family γ_i are disjoint,

3 any curve in a family γ_i has only one intersection with any curve in another family γ_j, for $i, j \in \{1, 2, 3\}$ distinct.

Note that in a complete 3-web $\{\gamma_1, \gamma_2, \gamma_3\}$, there exist exactly three different curves passing a given point in Σ'.

Now suppose there are three curves $L_i \in \gamma_i$ for $i = 1, 2, 3$ in a complete 3-web $\{\gamma_1, \gamma_2, \gamma_3\}$, such that $L_1 \cap L_2 \cap L_3 = \{\sigma\}$. For any point $\sigma' \neq \sigma$ on the curve L_i, we define a rule,

$$p_{ij} : L_i \to L_j, \tag{5}$$

such that a curve in γ_k passes both σ' and $p_{ij}(\sigma')$ for all distinct $i, j, k \in \{1, 2, 3\}$. Notice that $p_{ij} = p_{ji}^{-1}$, so $p_{ij} \circ p_{ji}$ is an identity function on L_i for all $j \neq i$.

Although $p_{ki} \circ p_{jk} \circ p_{ij}$ and $p_{ji} \circ p_{kj} \circ p_{ik}$ will both return back to a point in L_i, they are not necessarily identical. For instance, if we start from a point $\sigma_1 \in L_1$, then

$$p_{12}(\sigma_1) = \sigma_2 \in L_2, \quad p_{23}(\sigma_2) = \sigma_3 \in L_3, \quad p_{31}(\sigma_3) = \sigma_4 \in L_1,$$
$$p_{12}(\sigma_4) = \sigma_5 \in L_2, \quad p_{23}(\sigma_5) = \sigma_6 \in L_3, \quad p_{31}(\sigma_6) = \sigma_7 \in L_1.$$

We shall call the region shaped by the points $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7$ a *sequentially constructed hexagon*. If $\sigma_7 = \sigma_1$, such a hexagon is then called *closed*. In effect, we can see if $p_{31} \circ p_{23} \circ p_{12} = p_{21} \circ p_{32} \circ p_{13}$, then

$$p_{31} \circ p_{23} \circ p_{12} \circ p_{31} \circ p_{23} \circ p_{12} = p_{31} \circ p_{23} \circ p_{12} \circ (p_{21} \circ p_{32} \circ p_{13})^{-1}$$

will be an identity function, and thus any sequentially constructed hexagon will be closed. The next Fig. 2 shows a closed sequentially constructed hexagon on the plane which is embedded with a linear 3-web.

Definition 3.2. *A 3-web on Σ' is called hexagonal, if any sequentially constructed hexagon in Σ' is closed.*

Thomsen [12] proved that a planar 3-web is hexagonal if and only if it is equivalent to some linear 3-web on the plane. The Thomsen condition for a preference relation is proposed, largely because it is rather hard to geometrically identify equivalence conditions between different 3-webs before we know their algebraic representations.

Lemma 3.1. *Assume a preference relation $\gtrsim$ on Σ' admits a utility representation $u : \Sigma' \to \mathbb{R}$. Then $\gtrsim$ on Σ' satisfies the Thomsen condition, if and only if the 3-web $\{x, y, u(x, y)\}$ on Σ' is hexagonal.*

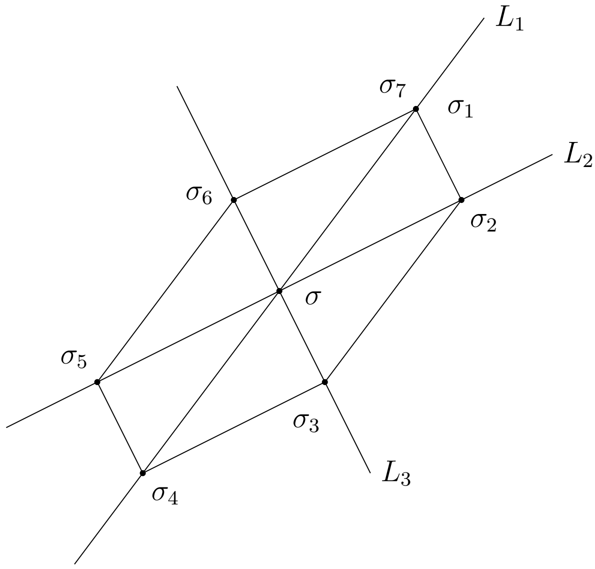

Fig. 2.: A sequentially constructed hexagon on the plane.

Let $\{x, y, w(x, y)\}$ be a complete 3-web on $\Sigma' = X \times Y$, where $w : \Sigma' \to \mathbb{R}$. Define a transformation operator $\bowtie$, such that for all (x_1, y_1) and (x_2, y_2) in Σ'

$$(x_1, y_1) \bowtie (x_2, y_2) = (x_1, y_2), \tag{6}$$

where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Note that (x_1, y_2) and (x_1, y_1) are on a same curve $x = x_1$ in the family x, and (x_1, y_2) and (x_2, y_2) are on a same curve $y = y_2$ in the family y, then we can set rules so that (x_1, y_1) on a curve in the family y moves to (x_1, y_2) on a curve in the family $w(x, y)$, and moves to (x_2, y_2) on a curve in the family x. Thus (x_1, y_1), (x_1, y_2), and (x_2, y_2) will be consecutive in a sequentially constructed hexagon on $\{x, y, w(x, y)\}$.

Proof of Lemma 3.1. We have three regular families of smooth curves,

$$g_1(x, y) = x, \quad g_2(x, y) = y, \quad g_3(x, y) = u(x, y),$$

where x and y are coordinate bases of Σ', and $u(x, y)$ is determined by the preference relation $\succsim$ on Σ', thus $\{x, y, u(x, y)\}$ is a complete 3-web on Σ'.

If $\{x, y, u(x, y)\}$ on Σ' is hexagonal, then we need to show that $(x_1, y_2) \sim (x_2, y_1)$ and $(x_2, y_3) \sim (x_3, y_2)$ can sufficiently imply $(x_1, y_3) \sim (x_3, y_1)$ for all $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Note that

$$(x_1, y_2) \bowtie (x_2, y_3) = (x_1, y_3), \quad (x_2, y_3) \sim (x_3, y_2),$$

and also

$$(x_3, y_2) \bowtie (x_2, y_1) = (x_3, y_1), \quad (x_2, y_1) \sim (x_1, y_2).$$

Thus the two sequences of consecutive points (x_1, y_2), (x_1, y_3), (x_2, y_3), (x_3, y_2), and (x_3, y_2), (x_3, y_1), (x_2, y_1), (x_1, y_2) are both in certain sequentially constructed hexagons. Since $\{x, y, u(x, y)\}$ is hexagonal, those two sequences should form a closed hexagon in Σ'. Since $(x_2, y_3) \sim (x_3, y_2)$ and $(x_2, y_1) \sim (x_1, y_2)$, both pairs of points should be on same curves in the family $u(x, y)$. Thus the remaining two points, (x_1, y_3) and (x_3, y_1), must be also on a same curve in $u(x, y)$, as any closed hexagon is shaped by 3 curves in x, 3 curves in y, and 3 curves in $u(x, y)$. It now appears that $(x_1, y_3) \sim (x_3, y_1)$, and thus the Thomsen condition is satisfied by $\gtrsim$ on Σ', which completes the proof of the sufficient part.

If $\gtrsim$ satisfies the Thomsen condition on Σ', then

$$(x_1, y_2) \sim (x_2, y_1), \quad (x_2, y_3) \sim (x_3, y_2), \quad (x_1, y_3) \sim (x_3, y_1),$$

for all $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. We again have

$$(x_1, y_2) \bowtie (x_2, y_3) = (x_1, y_3), \quad (x_3, y_2) \bowtie (x_2, y_1) = (x_3, y_1).$$

So the two sequences of (x_1, y_2), (x_1, y_3), (x_2, y_3), and (x_3, y_2), (x_3, y_1), (x_2, y_1) are both on sequentially constructed hexagons.

Note that (x_2, y_1) and (x_3, y_1) are on a same curve in y, and (x_2, y_1) and (x_1, y_2) are on a same curve in $u(x, y)$, so (x_3, y_1), (x_2, y_1), and (x_1, y_2) are consecutive points in a sequentially constructed hexagon. Similarly, (x_2, y_3), (x_3, y_2), and (x_3, y_1) are in a sequentially constructed hexagon as well. Thus the sequence of the seven points

$$(x_1, y_2), \quad (x_1, y_3), \quad (x_2, y_3), \quad (x_3, y_2), \quad (x_3, y_1), \quad (x_2, y_1), \quad (x_1, y_2)$$

form a closed sequentially constructed hexagon. Since x_1, x_2, x_3 and y_1, y_2, y_3 are all picked arbitrarily, the 3-web $\{x, y, u(x, y)\}$ must be hexagonal, which completes the proof of the necessary part. ∎

By applying the transformation operator $\bowtie$ on Σ', we can have a more intuitive illustration of the Thomsen condition.

Let $\sigma_1 = (x_1, y_2)$, $\sigma_2 = (x_2, y_1)$, $\sigma_3 = (x_2, y_3)$, and $\sigma_4 = (x_3, y_2)$. So σ_1 and σ_4 are on the same curve $y = y_2$ in the family y, and σ_2 and σ_3 are on the same curve $x = x_2$ in the family x. A preference relation $\gtrsim$ on Σ' satisfies the Thomsen condition, if $\sigma_1 \sim \sigma_2$ and $\sigma_3 \sim \sigma_4$ imply

$$\sigma_1 \bowtie \sigma_3 \sim \sigma_4 \bowtie \sigma_2,$$

where $\sigma_1 \bowtie \sigma_3 = (x_1, y_3)$ and $\sigma_4 \bowtie \sigma_2 = (x_3, y_1)$. Its graphic illustration has been shown in Fig. 3.

Note that

$$\sigma_3 \bowtie \sigma_1 = \sigma_2 \bowtie \sigma_4 = (x_2, y_2),$$

and in addition,

$$\sigma_1 \bowtie \sigma_4 = \sigma_1, \quad \sigma_3 \bowtie \sigma_2 = \sigma_2, \quad \sigma_4 \bowtie \sigma_1 = \sigma_4, \quad \sigma_2 \bowtie \sigma_3 = \sigma_3.$$

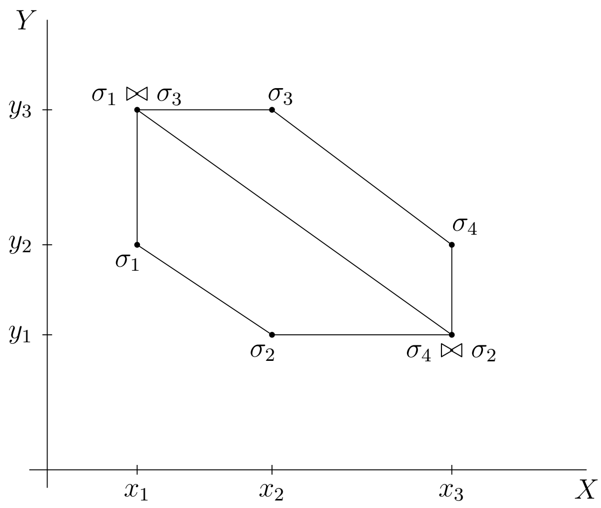

Fig. 3.: An alternative illustration of the Thomsen condition.

Let $\delta \in \{\sigma_1, \sigma_2\}$ and $\delta' \in \{\sigma_1, \sigma_2\} \setminus \{\delta\}$. Let $\kappa \in \{\sigma_3, \sigma_4\}$ and $\kappa' \in \{\sigma_3, \sigma_4\} \setminus \{\kappa\}$. We therefore have a more general invariant relation between those two indifference sets $\{\sigma_1, \sigma_2\}$ and $\{\sigma_3, \sigma_4\}$: for all δ and κ

$$\delta \bowtie \kappa \sim \kappa' \bowtie \delta', \tag{7}$$

by which the specific implication of the Thomsen condition can be clearly included.

4. NECESSARY CONDITION

The main result of this section is that the Thomsen condition on a general domain $\Sigma = X_1 \times X_2$, which is equivalent to $\Sigma' \subseteq \mathbb{R}^2$ through a diffeomorphism, is also necessary for a preference structure on Σ being additive. Thus a rational preference is additive on Σ can be totally captured by a pair of axioms, *videlicet*, the independence condition and the Thomsen condition.

Theorem 4.1. *If a preference relation $\gtrsim$ on $\Sigma' = X \times Y$ is additive, then $\gtrsim$ must satisfy the Thomsen condition.*

Note that any 3-web $\{\gamma_1, \gamma_2, \gamma_3\} \simeq \{x, y, u(x, y)\}$ on Σ' can represent a certain class of rational preferences on Σ', where $u : X \times Y \to \mathbb{R}$ serves as the utility function representing the family of indifference curves γ_3. A 3-web $\{x, y, u(x, y)\}$ is hexagonal, if and only if its curvature is zero (cf. Akivis and Goldberg [1], p. 207). Recall that the curvature of the 3-web $\{x, y, u(x, y)\}$ can be defined as

$$k(u) = -\frac{1}{u_x u_y} \frac{\partial^2}{\partial x \partial y} \log(u_x/u_y), \tag{8}$$

where $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$.

If $u(x, y)$ is additive, i.e. $u(x, y) = u_1(x) + u_2(y)$, then

$$u_x/u_y = u_1'(x)/u_2'(y),$$

where $u_1' = \mathrm{d}u_1/\mathrm{d}x$ and $u_2' = \mathrm{d}u_2/\mathrm{d}y$, and thus $k(u) = 0$. However, if $u(x, y)$ is just in general separable, i.e. $u(x, y) = v(u_1(x), u_2(y))$, where $v : \mathbb{R}^2 \to \mathbb{R}$, $k(u)$ will not be necessarily zero. For instance, consider

$$u(x, y) = u_1(x) + \log(u_1(x) + u_2(y)), \tag{9}$$

in which $u_x/u_y = (u_1 + u_2 + 1)u_1'/u_2'$, then clearly,

$$k(u) = \frac{(u_1 + u_2)^2}{(u_1 + u_2 + 1)^3} \neq 0. \tag{10}$$

Proof of Theorem 4.1. By the definition of additive preference relation, $\gtrsim$ on $\Sigma' = X \times Y$ admits at least one additive conjoint representation. Let $u(x, y) = u_1(x) + u_2(y)$ be such a utility representation, then $\{x, y, u_1(x) + u_2(y)\}$ is a 3-web on Σ'. Define a mapping g that maps any $(x, y) \in \Sigma'$ to $(u_1(x), u_2(y))$, then it appears that g is bijective. Let the image of g be $\Sigma^* = Z_1 \times Z_2$. We then have an equivalent 3-web $\{z_1, z_2, z_1 + z_2\}$ on $Z_1 \times Z_2$. The linear 3-web $\{z_1, z_2, z_1 + z_2\}$ is hexagonal, so is $\{x, y, u_1(x) + u_2(y)\}$ on Σ'. By Lemma 3.1, $\gtrsim$ satisfies the Thomsen condition on Σ', which completes the proof. ∎

Corollary 4.1. *If a preference relation $\gtrsim$ is additive on a domain Σ which could be transformed to an affine domain $\Sigma' \subseteq \mathbb{R}^2$ by some diffeomorphism, then $\gtrsim$ satisfies the Thomsen condition.*

Proof. Suppose there exists a diffeomorphism $f : \Sigma \to \mathbb{R}^2$, which transforms Σ to $\Sigma' = X \times Y$. The preference relation $\gtrsim$ will be again additive on Σ'. By Theorem 4.1, $\gtrsim$ satisfies the Thomsen condition on Σ', thus $\gtrsim$ also satisfies the Thomsen condition on Σ, as f is bijective between Σ and Σ'. ∎

As a result, if a rational preference relation $\gtrsim$ is independent on a general domain $\Sigma = X_1 \times X_2$, the Thomsen condition is not only sufficient but also necessary for its additive structure. By the discussion above based on the curvature, we should see that the Thomsen condition on Σ' is too strong for its separability. However, there still exists some general relation between the separability and the Thomsen condition. In effect, we will next show that a separable preference structure is additive if and only if its separability rule can represent an additive preference structure on a proper subset of $\mathbb{R}^2$.

Theorem 4.2. *A separable preference relation $\gtrsim$ on $\Sigma = X_1 \times X_2$ with a separability rule $v : \mathbb{R}^2 \to \mathbb{R}$ is additive, if and only if v can represent a preference relation on $u_1(X_1) \times u_2(X_2)$ satisfying the Thomsen condition, where $u_i : X_i \to \mathbb{R}$ for $i = 1, 2$.*

Proof. If $v : \mathbb{R}^2 \to \mathbb{R}$ can represent a preference relation $\succsim'$ on the domain $\Sigma' = u_1(X_1) \times u_2(X_2)$ satisfying the Thomsen condition, then by Debreu's theorem [3], $\succsim'$ must be additive on Σ'. Thus $\succsim'$ on Σ' admits an additive conjoint representation, $v_1(x) + v_2(y)$ for all $(x, y) \in \Sigma'$, where $v_i : u_i(X_i) \to \mathbb{R}$ for $i = 1, 2$. There then must be a positively monotonic function $g : \mathbb{R} \to \mathbb{R}$, such that $v(x, y) = g(v_1(x) + v_2(y))$ can also represent $\succsim'$ on Σ' for all $(x, y) \in \Sigma'$.

Note that for all $x = u_1(x_1)$ and $y = u_2(x_2)$, we have the utility representation for $\succsim$,

$$g(v_1 \circ u_1(x_1) + v_2 \circ u_2(x_2)) = v(u_1(x_1), u_2(x_2)),$$

where $(x_1, x_2) \in \Sigma$. Scaling $v(u_1(x_1), u_2(x_2))$ by the positively monotonic function g^{-1}, we obtain another utility representation for $\succsim$,

$$u(x_1, x_2) = v_1 \circ u_1(x_1) + v_2 \circ u_2(x_2),$$

which suggests that $\succsim$ is actually additive on Σ.

On the other hand, if $\succsim$ is separable on Σ with a separability rule v, then $\succsim$ should admit the specific utility representation $u(x_1, x_2) = v(u_1(x_1), u_2(x_2))$, where $(x_1, x_2) \in X_1 \times X_2$. Suppose v is not additive on $\Sigma' = u_1(X_1) \times u_2(X_2)$, then $v(x, y)$, where $(x, y) \in \Sigma'$, can not be any function like $v_1(x) + v_2(y)$ up to all positively monotonic transformation. But it would imply $\succsim$ on Σ can not admit any utility function like $v_1 \circ u_1(x_1) + v_2 \circ u_2(x_2)$ up to all positively monotonic transformation, which means $\succsim$ can not be additive, a contradiction. Therefore, v must be additive on $u_1(X_1) \times u_2(X_2)$, and by Theorem 4.1, it should satisfy the Thomsen condition. ∎

Example 4.1. *Consider the Cobb-Douglas utility function* $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$ *on the domain* $\Sigma = \mathbb{R}^+ \times \mathbb{R}^+$, *where* $\alpha, \beta \in (0, 1)$. *It can be equivalently expressed as*

$$u(x_1, x_2) = \exp(\alpha \log x_1 + \beta \log x_2), \tag{11}$$

where $(x_1, x_2) \in \Sigma$. *Note that* $\exp(x + y)$ *is additive on* $\Sigma' = \log(X_1) \times \log(X_2)$, *as it is equivalent to* $x + y$ *on* Σ' *by the positively monotonic transformation* "log". *Thus, the Cobb-Douglas utility function represents an additive preference relation on* Σ. *Evidently, we know it is equivalent to the utility function* $\alpha \log x_1 + \beta \log x_2$.

Example 4.2. *Consider a utility function*

$$u(x_1, x_2) = \min\{\alpha u_1(x_1), \beta u_2(x_2)\} \tag{12}$$

on a general domain $\Sigma = X_1 \times X_2$, *where* $\alpha, \beta \neq 0$, *and* $u_i : X_i \to \mathbb{R}$ *for* $i = 1, 2$. *It is a representation of the Leontief class of utility functions. We can observe that u is separable with a separability rule* "min". *However, it is not additive, as in any case* $\min\{x, y\}$ *can not be additive on* $u_1(X_1) \times u_2(X_2)$.

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