A THEORY OF LIQUIDITY MEASURE IN MODERN SECURITY MARKETS

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Abstract. The present paper constructs a unified and concise theory of liquidity measure in modern security markets. The market conditions used in our theory simply focus on the best quotations and their depths, which are both easily available in a fast and electronic security market. Two liquidity measures are then defined to capture the four liquidity dimensions, *viz.*, the width, depth, immediacy, and resiliency. In general, a pure liquidity measure can reflect the width, immediacy, and resiliency in certain functional forms, while a proper liquidity measure depends on all the four liquidity dimensions, and sometimes it is separable in its pure quotation dimension and depth dimension.

1. Introduction

Liquidity is a quite important concept in both theoretical and empirical studies of modern financial markets, and its characterization is usually thought of as a basic index for market quality. Typically, a liquid market is characterized by a number of attributes which have been commonly accepted by researchers and analysts. For example, Demsetz (1968) associated liquidity with "the cost of making transactions without delay", in other words, the immediacy. Black (1971) described a liquid market by a combination of its facets, that is, a continuous market (in more recent words, trading on the market being immediate), and an efficient market which can furthermore be recognized as that the bid-ask spread is small, the market depth is substantial, and the quotation is resilient. Kyle (1985) referred to three transactional properties of a market, that is, the tightness, depth, and resiliency. Similarly, Bernstein (1987) and also Schwartz and Francioni (2004, Chapter 3) focused on the depth, breadth (or tightness), and resiliency of a market. Massimb and Phelps (1994) refers to such market abilities as the immediacy and resiliency. More recently, Harris (2003, Chapter 19) proposed four liquidity dimensions, *viz.*, the immediacy, width, depth, and resiliency. That being all said, there does not exist a unified formal definition of liquidity. As far as it is concerned, the very essence of liquidity should be related

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to the transaction cost, or more precisely, the cost of immediate execution. Clearly, a higher cost of immediate execution would lead to a more illiquid (or inversely speaking, less liquid) market.

Because of lacking a unified formal definition, liquidity measures adopted in financial studies and analytical activities are very diverse, which partly depend on market structures, say highly active markets like equity or security markets, or inactive ones like the bond market. In a highly active security market, the liquidity measure is more easily constructed based on trading activities on the market. There are a lot of such liquidity proxies, for example, the parameter λ of Kyle (1985) that relies on the price impact of trading, the illiquidity ratio ILLIQ of Amihud (2002) that uses the absolute return and trading volume to give the price impact of order flow, and the Amivest measure proposed by Amivest Capital Management, which as a liquidity ratio is very similar to ILLIQ. These above liquidity measures[1] all show that a higher price impact implies a lower liquidity state. With regards the not-so-active bond market, turnover is accepted as an implicit measure of liquidity state, for example, the latent liquidity measure proposed by Mahanti *et al.* (2008) does not use any transaction data which are very sparse in that market, but be just calculated from the turnover of bond investors.

From a different perspective, the following three categories of liquidity measure might be suggested to show more clearly its diversity:

(i) measures based on the nature of liquidity, *viz.*, the transaction cost,
(ii) measures constructed out of market conditions,
(iii) measures depending on cause-and-effect empirical analysis.

We might very briefly demonstrate some examples. The effective spread, the quoted spread, the market depth, and the probability of informed trading clearly belong to category (ii). The price impact of trading or order flow as was mentioned above is a popular proxy in category (iii). Since transaction costs consist of explicit and implicit parts, and it is hard to catch implicit execution costs, liquidity measures in category (i) would frequently have their operational meanings conveyed by these in category (ii) or (iii).

In this study, we plan to propose a general mathematical definition of liquidity measure, so that most practical measures would become special cases under our definition. It could thus help lay some rigorous foundations for the notion of liquidity, and place it in a more precise analytic framework.

The present paper is organized as follows. Section 2 introduces the structure of a modern security market, in which the order book is characterized two-dimensionally

[1]Schwartz and Francioni (2004, Chapter 3) argued that the (il)liquidity ratio is in effect a common misconception, as trading is not only triggered by idiosyncratic factors, but also information changes which in a sense are linked with the market efficiency.

by the quotation and the depth. Two concepts, the market state and the shape function, are formally developed in their abstract forms, which would be useful for the pure liquidity measure in Section 3 and the proper liquidity measure in Section 4. Section 2 also discusses in detail the liquidity state, and describes the reasoning procedure for reducing its determinant to the key information on best quotations and volumes standing there. Section 3 and 4 comprise the most significant parts of this paper, in which two liquidity measures, pure and proper, are formally defined. The last section concludes our investigations.

2. Market Structure

We first introduce the market structure of a generic modern security market, which is typically order-driven. Let the best bid, best ask, bid-ask spread, and midprice on the market be b, a, s, and m, respectively. We can directly write

$$s = a - b, \quad m = (b + a)/2.$$

Let (b, a), or more concisely w, denote the bid-ask pair. Note that $s \geq \underline{s}$, $a \leq \overline{a}$, and $b \geq 0$, where $\underline{s} > 0$ is the lower bound of the bid-ask spread, and $\overline{a}$ is the upper bound of the ask, so the bid-ask domain can be expressed as

$$W = \{(b, a) \in \mathbb{R}^2 : a - b \geq \underline{s},\ a \leq \overline{a},\ \text{and } b \geq 0\},$$

which admits of a geometric interpretation, *viz.*, W is a triangle with vertices $(0, \underline{s})$, $(0, \overline{a})$, and $(\overline{a} - \underline{s}, \overline{a})$ in the w-plane (see Fig. 1).

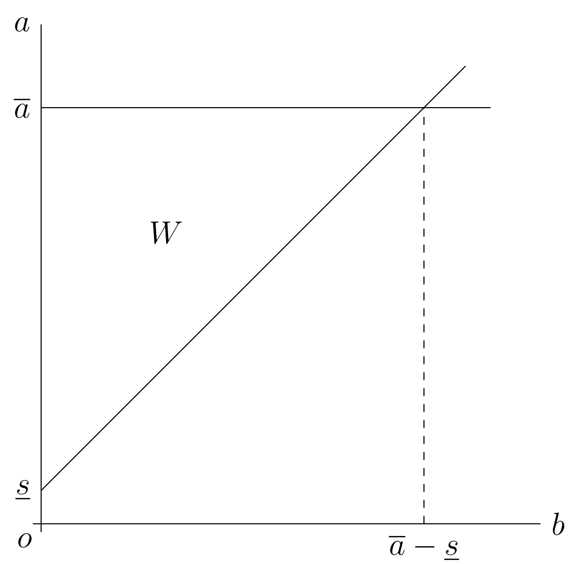

Fig. 1

Define the spread function $s : W \to \mathbb{R}^+$, and the midprice function $m : W \to \mathbb{R}^+$, such that for all $w = (b, a)$

$$s(w) = a - b, \quad m(w) = (b + a)/2.$$

More essential information on the market is effectively carried by its functioning order book, which includes dispersed quotes on both bid and ask sides, and the aggregated trading depth at each trading commitment. As we have defined, the bid and ask centered on the order book are b and a, respectively. Suppose all the other less attractive bid quotations on the bid side are $b_1, b_2, \dots$, and all the less attractive ask quotations on the ask side are $a_1, a_2, \dots$, so that they are discretely distributed over $[0, \overline{a}]$ as follows:

$$0 \le \cdots < b_2 < b_1 < b < a < a_1 < a_2 < \cdots \le \overline{a}.$$

Any quote existing in the order book as a trading commitment should be associated with a nonzero trading depth, and usually the trading depth varies across the whole book, simply as trading volumes (or shares) at different quotes are frequently unequal. At any time, the order book should then have a certain distribution of shares, which can be completely determined by its quote dispersion, that's to say, the trading depth is actually a function in the quote. And after some trading arrangements executed by the market, the original distribution would be updated accordingly, and very likely with some additional perturbation coming from contagious noise trading arrangements.

Let q denote a generic quote existing in the order book, then we should see $q \in [0, \overline{a}]$. Notice that the nonempty open interval (b, a) is never available for a market order, because there are no shares at all. Let β, α, and σ be the state that $q \in [0, b]$, $q \in [a, \overline{a}]$, and $q \in (b, a)$, respectively. Let $P = [0, \overline{a}]$, then it can be partitioned into

$$P_\beta = [0, b], \quad P_\alpha = [a, \overline{a}], \quad P_\sigma = (b, a).$$

Evidently, $P = P_\beta \cup P_\alpha \cup P_\sigma$, in which P_β, P_α, and P_σ are pairwise disjoint (see Fig. 2).

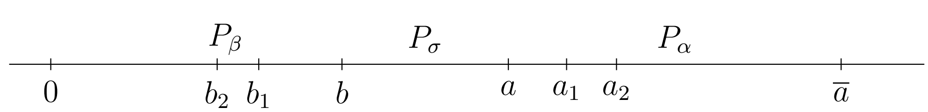

Fig. 2

Definition 1. A function $f : P \to \mathbb{R}^+$ is called a *shape function*, if $f(q) = 0$ for all $q \in P_\sigma$, and $f(q) \neq 0$ for all $q \in P_\beta \cup P_\alpha$.

Note that a specific market condition now can be described by the dispersed quotes and the shape function of the order book. By separating these two aspects of market condition, we would propose two distinct, but closely connected, approaches to look into its liquidity state. Concretely, we first only consider the quote dispersion, so that the attained liquidity measure does not cover the depth dimension, but only cover the immediacy, width, and resiliency dimensions; we next take account of both aspects of market condition, and thus the new liquidity measure should be more comprehensive to reflect all the four dimensions.

As we have so far clearly defined, at any time the quote dispersion on the order book can be represented by a finite long-dimensional tuple

$$(\dots, b_2, b_1, b, a, a_1, a_2, \dots),$$

and accordingly, there should be a deterministic shape function $f(q)$, such that $f(q) > 0$ for all q appearing in the above tuple. In an over-generalized sense, the liquidity state of an order book should be determined by all the quotes with a nonzero depth. But if one realizes that the bid-ask pair (b, a) already carries enough information on the market, as the quote dispersion on the bid and ask sides are both statistically related to the bid-ask spread (see for example, Biais, Hillion, and Spatt (1995)), then a function of bid-ask pair should be a quite reasonable treatment of liquidity measure.

In the next sections, we shall call a measure of liquidity on the width, immediacy, and resiliency dimensions a pure liquidity measure, and call a measure of liquidity on all the four dimensions, *viz.*, the width, immediacy, resiliency, and depth, a proper liquidity measure.

3. Pure Measure

Definition 2. $\rho : W \to [0, 1]$ is a *pure liquidity measure*, if there exist $\psi : W \to \mathbb{R}$ and $\phi : \mathbb{R} \to [0, 1]$ such that $\rho = \phi \circ \psi$, for ϕ inconstant and ψ nondegenerate.

In other words, if ρ is a pure liquidity measure, then there should be a family of curves on W defined by some ψ, such that any bid-ask pair on a same curve in the family must have a same liquidity state determined by $\phi = \rho \circ \psi^{-1}$.

Example 1. Let $\psi(w) = s(w)$. Recall that $s(w) = a - b$ for all $w = (b, a)$, so $s(w) \in [\underline{s}, \overline{a}]$ for all $w \in W$. Notice that the bid-ask spread s is usually considered as an absolute index for the width aspect of the order book, thus we might use its normalized value to represent the tightness, *viz.*, $(\overline{a} - s)/(\overline{a} - \underline{s})$, which then linearly varies from 1 (with a spread $\underline{s}$) to 0 (with a spread $\overline{a}$).

The immediacy aspect of the order book is actually an alternative perception to its capacity for market orders, which should be negatively related to the log-scaled spread, as was shown by Wang (2013), thus we assume that the immediacy for a spread s could also be represented by a normalized value, $(\log \overline{a} - \log s)/(\log \overline{a} - \log \underline{s})$.

With regards the resiliency aspect of the order book, we rely on the multiplication of tightness and immediacy to generate a representation for such a liquidity dimension. Consequently, the function $\phi : [\underline{s}, \overline{a}] \to [0, 1]$ is now defined to be

$$\phi(s) = \frac{(\overline{a} - s)(\log \overline{a} - \log s)}{(\overline{a} - \underline{s})(\log \overline{a} - \log \underline{s})}, \tag{1}$$

and thus we obtain a pure liquidity measure ρ that is constructed by $\phi \circ \psi$,

$$\rho(w) = \frac{\big(\overline{a} - s(w)\big)\big(\log \overline{a} - \log s(w)\big)}{(\overline{a} - \underline{s})(\log \overline{a} - \log \underline{s})}. \tag{2}$$

Set $\overline{a}/\underline{s} = r$ and $s/\underline{s} = s_d$, so $s_d \in [1, r]$. Since $\overline{a} \gg \underline{s}$, we have $r \gg 1$, and hence $r - 1 \approx r$. Then (1) can be rewritten as a function in s_d,

$$\phi(s_d) \approx \Big(1 - \frac{s_d}{r}\Big)\Big(1 - \frac{\log s_d}{\log r}\Big) = 1 - \frac{s_d}{r} - \frac{\log s_d}{\log r} + \frac{s_d \log s_d}{r \log r}. \tag{1'}$$

Let's redefine $\psi(w) = s_d(w) = s(w)/\underline{s}$, then the pure liquidity measure expressed by (2) has a finer expression,

$$\rho(w) = 1 - \frac{s_d(w)}{r} - \frac{\log s_d(w)}{\log r} + \frac{s_d(w) \log s_d(w)}{r \log r}. \tag{2'}$$

And we should see more clearly the roles of $s_d(w)$, $\log s_d(w)$, and their multiplication respectively for the tightness, immediacy, and resiliency dimension in the pure liquidity measure $\rho(w)$. Note that $-s_d$ and $-\log s_d$ are both decreasing with s_d, while $s_d \log s_d$ is increasing with s_d so that it can balance against the former decreasing rates.

Example 2. Let $\psi(w) = s_r(w) = \frac{s(w)}{2m(w)}$. At any $w = (b, a)$, we have the spread $s = a - b$, the midprice $m = (b + a)/2$, and the relative spread s_r,

$$s_r = \frac{s}{2m} = \frac{a - b}{a + b}.$$

Evidently, $s_r \in [\underline{s}/(2\overline{a} - \underline{s}), 1]$ for all $w \in W$, and approximately stating, it is nearly $(0, 1]$ simply as $\overline{a} \gg \underline{s}$.

Let $\phi(s_r) = -e s_r \log s_r$, where $e \approx 2.718$. Note that $-s_r \log s_r$ has the maximum of e^{-1} and minimum of 0 in the domain $(0, 1]$, so $\phi(s_r) \in [0, 1]$ for all s_r. In this situation, we have another pure liquidity measure $\varrho(w) = \phi\big(s_r(w)\big)$, and for all $w = (b, a)$

$$\varrho(w) = -e \frac{a - b}{a + b} \log\Big(\frac{a - b}{a + b}\Big). \tag{3}$$

So $\varrho(w)$ achieves the maximum of 1, when $s_r(w) = e^{-1}$, or $b/a = (e-1)/(e+1)$, which is roughly 0.462. It might be noticed that

$$\frac{e-1}{e+1} = \frac{\sinh(1/2)}{\cosh(1/2)} = \tanh(1/2),$$

where "cosh" and "sinh" are hyperbolic cosine and sine, respectively. We can thus state, in a much conciser way, that $\varrho(w)$ has its maximum if $b/a = \tanh(1/2)$.[2] The following Fig. 3 shows some geometric properties of the optimal liquidity states on the (b, a)-plane, in which the angle $\vartheta = \arctan e^{-1} \approx 0.353$, or about 20.20° in degree measure.

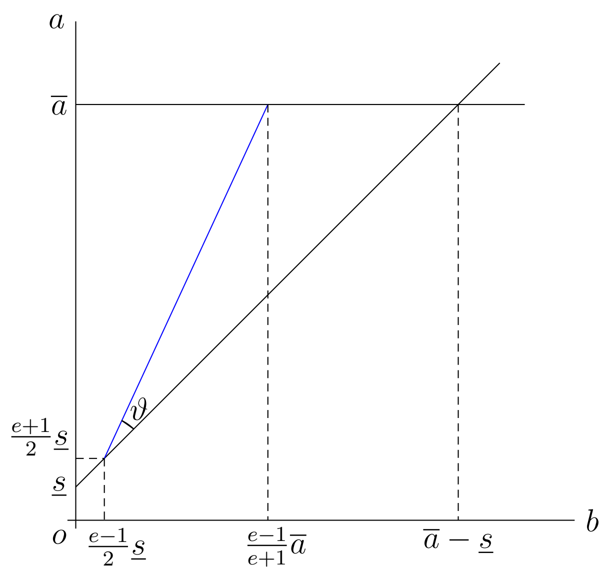

Fig. 3

4. Proper Measure

Let δ_f denote the market volume in a time when the shape function is $f(q)$, with $f(q) > 0$ for all $q \in P_\beta \cup P_\alpha$, then we have

$$\delta_f = \int_P f(q)\,\mathrm{d}q = \sum_{f(q)>0} f(q). \tag{4}$$

[2] As a general conjecture, there must be some real function for any pure liquidity measure, say ϖ, such that the so-measured liquidity would be maximized if $b/a = \varpi(1/2)$.

If, instead of being plainly aggregated, the trading volumes standing on the market are consolidated in a nonlinear way, then there is a weighting function[3] $\pi : P \to [0,1]$, such that the market volume is now expressed as

$$\delta_f = \int_P \pi(q) f(q)\,\mathrm{d}q = \sum_{f(q)>0} \pi(q) f(q). \tag{4'}$$

As a rather particular case, suppose, for some regulation reason, that $\pi(q) = 0$ if $q \neq b, a$, then the weighted market volume only covers the depths at the two best quotes, and it is equal to

$$\pi(b)f(b) + \pi(a)f(a).$$

Definition 3. $\ell : W \times \mathbb{R}^+ \to [0,1]$ is a *proper liquidity measure*, if there exist a function $\chi : [0,1] \times \mathbb{R}^+ \to [0,1]$ and some pure liquidity measure $\rho : W \to [0,1]$, such that for all $w \in W$ and $\delta_f \in \mathbb{R}^+$

$$\ell(w, \delta_f) = \chi\big(\rho(w), \delta_f\big). \tag{5}$$

It clearly suggests that a proper liquidity measure essentially reflects its intrinsic pure liquidity state as well as the market depth. Very often, the function χ should increase with both $\rho(w)$ and δ_f, and the corresponding ℓ should satisfy

$$\lim_{\rho(w)\downarrow 0} \ell(w, \delta_f) = 0, \quad \lim_{\delta_f \downarrow 0} \ell(w, \delta_f) = 0.$$

So the pure liquidity state and the depth are both critical elements in a proper liquidity measure, because the liquidity as is properly measured would vanish when either of them goes to zero.

Example 3. Suppose $\chi\big(\rho(w), \delta_f\big)$ is separable in such a way that

$$\ell(w, \delta_f) = \rho(w)\big(1 - \exp(-\gamma\delta_f)\big), \tag{6}$$

where $\gamma > 0$, and $\rho(w)$ is a pure liquidity measure.

In this situation, the proper liquidity maximization can be processed as two separate targets, *viz.*, the maximization of pure liquidity and of market depth. Notice that when $\gamma\delta_f \geq 6.908$, one has

$$1 - \exp\big(-\gamma\delta_f\big) > 0.999,$$

thus the market depth would be very closely maximized if $\delta_f = 6.908/\gamma$. And the pure liquidity maximization would be achieved, if w is located on a certain curve

[3]It might illustrate the fact that some trading volumes are actually hidden to traders, or that trading volumes at different quotes should have differential levels of significance.

bounded by W on the w-plane. For instance, assume $\rho(w) = \frac{s(w)}{2m(w)}$ as defined by (3), then such a curve appears to be the following line segment bounded by W,

$$\{w \in W : b - \tanh(1/2)a = 0\}.$$

In case the shape function $f(q)$ is stable in a time period T, and δ_f is calibrated to cover only the π-weighted depths at the best quotes, then δ_f is actually a function in w,

$$\delta(w) = \pi(b)f(b) + \pi(a)f(a),$$

where $\pi(b)+\pi(a) = 1$. Clearly, $\delta(w)$ is additive in b and a due to $\pi \cdot f$. In consequence, the proper liquidity measure ℓ effective in T could be reduced to a new measure ℓ^* just in the bid-ask pair w, which can now be written as

$$\ell^*(w) = \chi\big(\rho(w), \delta(w)\big). \tag{7}$$

Once we have fixed the functions $\psi(w), \delta(w)$, and the operators ϕ, χ, the corresponding pure and proper measures $\rho(w)$ and $\ell^*(w)$ are well defined. Quotes of a modern security market on most stock exchanges are available through some public data aggregators, say Yahoo Finance, which provides real-time quotes for securities on NASDAQ and NYSE, and quotes with some delay (from 10 to 30 minutes) for securities on other exchanges in the world. Therefore, the measures can be implemented by collecting the time series data on the best quotes b and a. As for the measure $\ell(w, \delta_f)$, more data on the volumes at the best quotes are required, which are also available without difficulty.

It might depend on one's attitude towards data to select ℓ^* or ℓ in practice. If one thinks that a greater data dimension could bring on more inconsistencies, then it would be better to employ ℓ^*. But if one believes a higher theoretical setting could carry less empirical meanings, ℓ would then be a much more suitable choice.

5. Summary

This paper constructs two correlated liquidity measures called pure liquidity measure and proper liquidity measure to propose more precise meanings for the concept of liquidity in a mathematical way. We have presented a number of specific examples to show how our definitions can be applied to generate some conventional liquidity measures or proxies, which are widely adopted in financial studies and analytical activities. Concerning some normal liquidity measures, it seems that there always exists a set of market conditions which can hold a maximal liquidity level under a certain market structure.

In this study, the four liquidity dimensions have been divided into two classes, that is, the pure quotation dimension and the volume dimension. The pure quotation dimension which consists of the width, immediacy, and resiliency, outlines the construction of the pure liquidity measure, and the volume dimension which includes the depth information, adds itself to the pure liquidity measure to produce a proper one. Very often, the pure quotation dimension and volume dimension could be thought of to be separable, so that the proper liquidity measure would have some plain computing procedure to be lifted from its corresponding pure liquidity measure.

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